4+1 Formalism for a Local Metric with Parameterized Evolution

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Stueckelberg-Horwitz-Piron (SHP) Formalism

Electrodynamics		**Relativity**

Maxwell's equations

$\qquad\xrightarrow{\text{Einstein}}$

Lorentz group

Manifest covariance

$\xleftarrow{}$
Minkowski & Fock

Tensor formulation

$$\partial_\nu F^{\mu\nu}(x) = eJ^\mu(x)$$

$$\dot{x}^\mu = dx^\mu/ds$$

$$M\ddot{x}^\mu = eF^{\mu\nu}(x)\dot{x}_\nu$$

$$ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$$

Pair creation/annihilation

$\qquad\xrightarrow{\text{Stueckelberg}}$

External evolution parameter

$$E = p^0 = M\dot{x}^0 < 0$$

$$\dot{x}^\mu = dx^\mu/d\tau$$
$$ds^2 = -\dot{x}^2(\tau)d\tau^2$$

Covariant canonical dynamics

$\xleftarrow{}$
Horwitz & Piron

Unconstrained phase space

$$K = \frac{[p - eA(x)]^2}{2M} + V(x)$$

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

τ-dependent gauge field

$\qquad\xrightarrow{}$
Horwitz, Sa'ad,
Arshansky, Land

Evolving block universe

$$A_\mu(x) \longrightarrow a_\mu(x, \tau)$$

$$\mathcal{M} \to \mathcal{M}(\tau)$$

$$V(x) \longrightarrow -ea_5(x, \tau)$$

$$\frac{d}{d\tau}(\text{particle} + \text{field masses}) = 0$$

Stueckelberg-Horwitz-Piron (SHP) Formalism

Covariant canonical mechanics with parameterized evolution

8D unconstrained phase space $\implies \tau \neq$ proper time

$$x^\mu(\tau), \; \dot{x}^\mu(\tau) \qquad\qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad\qquad \lambda, \mu, \nu, \ldots = 0, 1, 2, 3$$

Canonical electrodynamics with scalar Hamiltonian ($K = $ total mass)

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e \dot{x}^\mu a_\mu(x, \tau) + e a_5(x, \tau) \qquad 0 = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}$$

$$K = \frac{[p - ea(x, \tau)]^2}{2M} - e a_5(x, \tau) \qquad\qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

Extend SHP to include pseudo-5D metric describing τ-evolution of spacetime

$$L = \frac{1}{2} M g_{\alpha\beta}(x, \tau) \dot{x}^\alpha \dot{x}^\beta \qquad \alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5 \qquad x^5 = c_5 \tau$$

Physical event $x^\mu(\tau)$ in SHP

Irreversible occurrence **at** time τ

$$\tau_2 > \tau_1 \implies \begin{cases} x^\mu(\tau_2) \text{ occurs } \textbf{after } x^\mu(\tau_1) \\ x^\mu(\tau_2) \textbf{ cannot change } x^\mu(\tau_1) \\ \text{No grandfather paradox} \end{cases}$$

Evolution

4D block universe $\mathcal{M}(\tau)$ **occurs** at τ

Infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occurs at $\tau + d\tau$

$$\mathcal{M}(\tau) \xrightarrow[\text{Hamiltonian } K \text{ generates evolution in } \tau]{} \mathcal{M}(\tau + d\tau)$$

$$\left. \begin{array}{c} \text{scalar } K \\ \text{external } \tau \end{array} \right\} \implies \text{No conflict with general diffeomorphism invariance}$$

Geometry and Trajectory

Standard approach to motion in general relativity

Two neighboring events in spacetime manifold $\mathcal{M}$ (instantaneous displacement)

Interval $\delta x^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu = (x_2 - x_1)^2$

Invariance of interval — geometrical statement about $\mathcal{M}$

Trajectory

Map arbitrary parameter ζ to sequence of events $x^\mu(\zeta)$

Timelike interval between any two events $\Rightarrow$ take $\zeta \to s$ (proper time)

In 4D block universe $\mathcal{M}$

Trajectory — sequence of instantaneous timelike displacements

"Motion" appears as displacements in $x^0(s)$

Path length $\longrightarrow$ Lagrangian

$$\delta x^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \delta s^2 \qquad\qquad g = (-,+,+,+)$$

$$L_{\text{constrained}} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \qquad\qquad L_{\text{unconstrained}} = \tfrac{1}{2}\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

Geometry and Evolution
Pseudo-5D metric

Two neighboring events:

$$x^\mu(\tau) \in \mathcal{M}(\tau) \qquad \bar{x}^\mu(\tau + \delta\tau) \in \mathcal{M}(\tau + \delta\tau)$$

Distance

$$dx^\mu = \bar{x}^\mu(\tau + \delta\tau) - x^\mu(\tau) \simeq \bar{x}^\mu(\tau) + \dot{\bar{x}}^\mu(\tau)\delta\tau - x^\mu(\tau) = \delta x^\mu + \dot{\bar{x}}^\mu \delta\tau$$

Squared interval (referred to x coordinates)

$$dx^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu + g_{\mu\nu}\dot{\bar{x}}^\mu \delta x^\nu \delta\tau + g_{\mu\nu}\dot{\bar{x}}^\mu \dot{\bar{x}}^\nu \delta\tau^2 \simeq \underbrace{g_{\alpha\beta}\left(x, \tau\right) \delta x^\alpha \delta x^\beta}_{\alpha,\beta=0,1,2,3,5}$$

Contributions to interval

$g_{\mu\nu}\,\delta x^\mu \delta x^\nu$

   Geometrical interval between two events at τ

   Expresses symmetries of spacetime manifold $\mathcal{M}$

$g_{55}\,\delta x^5 \delta x^5$

   Dynamical interval between events in $\mathcal{M}(\tau)$ and $\mathcal{M}(\tau + \delta\tau)$

   Expresses symmetries of evolution generated by Hamiltonian K

Example in space
Particle in 2D space — expanding disk with radius $R(\tau) = \frac{1}{2}g\tau^2$

Points on expanding disk

$$\mathbf{q} = R(\tau)(\cos\theta, \sin\theta) \qquad \bar{\mathbf{q}} = \bar{R}(\tau + \delta\tau)(\cos\bar{\theta}, \sin\bar{\theta})$$

Distance

$$d\mathbf{q} = \bar{\mathbf{q}} - \mathbf{q} \approx (\delta R + \delta_\tau R(\tau))\hat{\mathbf{R}} + R\delta\theta\,\hat{\boldsymbol{\theta}}$$

Geometrical distances: $\delta\theta = \bar{\theta} - \theta$ $\qquad \delta R = \bar{R}(\tau) - R(\tau)$

Dynamical distance: $\delta_\tau R(\tau) = R(\tau + \delta\tau) - R(\tau) = g\tau\delta\tau$

Interval

$$d\mathbf{q}^2 = \delta R^2 + R^2\delta\theta^2 + g^2\tau^2\delta\tau^2 + 2g\tau\delta R\delta\tau = g_{ab}\delta\zeta^a\delta\zeta^b$$

Pseudo-3D metric

$$\delta\zeta = (\delta R, \delta\theta, \delta\tau) \qquad g_{ab} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & R^2 & 0 \\ 1 & 0 & g\tau \end{pmatrix}$$

Example in space
Equations of motion

Lagrangian

$$L = \tfrac{1}{2} M g_{ab}\dot{\zeta}^a\dot{\zeta}^b = \tfrac{1}{2} M \left(\dot{R}^2 + R^2\dot{\theta}^2 + 2g\tau\dot{R} + g^2\tau^2\right)$$

$$\left.\begin{array}{l} 0 = \dfrac{d}{d\tau}M\left(\dot{R} + g\tau\right) - MR\dot{\theta}^2 \\[2ex] 0 = \dfrac{d}{d\tau}\left(MR^2\dot{\theta}\right) \longrightarrow MR^2\dot{\theta} = \ell \end{array}\right\} \longrightarrow M\ddot{R} = \dfrac{\ell^2}{MR^3} - Mg$$

Qualitative result

Particle at edge of disk sees force $F = \ell^2/MR^3 - Mg$

$Mg > \ell^2/MR^3 \implies$ Particle moves at edge of disk

                   As if attracted by gravitational force

$F_R = -Mg \longrightarrow$ Appears as "external" force

                   Enters through evolution of circular geometry

Canonical Mechanics in General 5D Spacetime

Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta}(x^\mu, x^5) \dot{x}^\alpha \dot{x}^\beta \qquad \lambda, \mu, \nu = 0, 1, 2, 3 \qquad \alpha, \beta, \gamma = 0, 1, 2, 3, 5$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha} \right)$$

Canonical momentum

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = M g_{\alpha\beta} \dot{x}^\beta \qquad \longrightarrow \qquad \dot{x}^\alpha = \frac{1}{M} g^{\alpha\beta} p_\beta$$

Conserved Hamiltonian

$$K = \dot{x}^\alpha p_\alpha - L = \frac{1}{2M} g^{\alpha\beta} p_\alpha p_\beta \qquad \frac{dK}{d\tau} = M g_{\alpha\beta} \dot{x}^\alpha \frac{D\dot{x}^\beta}{D\tau} = 0$$

Poisson bracket

$$\frac{dK}{d\tau} = \{K, K\} + \frac{\partial K}{\partial \tau} = \frac{1}{2M} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial \tau} = 0$$

Break 5D symmetry $\longrightarrow$ 4D+1

Constrain non-dynamical scalar $x^5 \equiv c_5\tau$

$$L = \tfrac{1}{2}M g_{\alpha\beta}(x,\tau)\dot{x}^\alpha \dot{x}^\beta = \tfrac{1}{2}M g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu + M c_5\,g_{\mu 5}\dot{x}^\mu + \tfrac{1}{2}M c_5^2\,g_{55}$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\alpha}{D\tau} = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma \longrightarrow \begin{cases} \ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma}\dot{x}^\lambda \dot{x}^\sigma + 2c_5\Gamma^\mu_{5\sigma}\dot{x}^\sigma + c_5^2\Gamma^\mu_{55} = 0 \\[2mm] \ddot{x}^5 = \dot{c}_5 \equiv 0 \end{cases}$$

Symmetry-broken connection

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}g^{\mu\nu}\left(\partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta}\right) \qquad\qquad \Gamma^5_{\alpha\beta} \equiv 0$$

Hamiltonian

$$K = p_\mu \dot{x}^\mu - L = \frac{1}{2M}g^{\mu\nu}p_\mu p_\nu - \tfrac{1}{2}M c_5^2\,g_{55}$$

$$\frac{dK}{d\tau} = \frac{\partial K}{\partial \tau} = -\tfrac{1}{2}M \dot{x}^\mu \dot{x}^\nu\,\partial_\tau g_{\mu\nu} - \tfrac{1}{2}M c_5^2\,\partial_\tau g_{55}$$

Matter

Non-thermodynamic dust

Number of events per spacetime volume $= n(x, \tau)$

Particle mass density $= \rho(x, \tau) = M n(x, \tau)$

5-component event current $= j^{\alpha}(x, \tau) = \rho(x, \tau) \dot{x}^{\alpha}(\tau) = M n(x, \tau) \dot{x}^{\alpha}(\tau)$

Continuity equation

$$\boldsymbol{\nabla}_{\alpha} j^{\alpha} = \boldsymbol{\nabla}_{\mu} j^{\mu} + \partial_{\tau} \rho$$

Mass-energy-momentum tensor

$$\boldsymbol{\nabla}_{\beta} T^{\alpha\beta} = 0 \qquad T^{\alpha\beta} = \rho \dot{x}^{\alpha} \dot{x}^{\beta} \longrightarrow \begin{cases} T^{\mu\nu} = \rho \dot{x}^{\mu} \dot{x}^{\nu} \\ T^{5\beta} = c_5 j^{\beta} \end{cases}$$

Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Weak Field Approximation

Small perturbation to flat metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \left(h_{\alpha\beta} \right)^2 \approx 0 \qquad h \simeq \eta^{\alpha\beta} h_{\alpha\beta}$$

Define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \longrightarrow$ Einstein equations

$$\frac{16\pi G}{c^4} T_{\alpha\beta} = \partial_\beta \partial_\gamma \bar{h}_\alpha^\gamma + \partial_\alpha \partial_\gamma \bar{h}_\beta^\gamma - \partial_\gamma \partial_\gamma \bar{h}_{\alpha\beta} - \partial_\alpha \partial_\beta \bar{h}$$

Impose gauge condition $\partial_\lambda \bar{h}^{\alpha\lambda} = 0 \longrightarrow$ wave equation

$$\frac{16\pi G}{c^4} T_{\alpha\beta} = -\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} = -\left(\partial^\mu \partial_\mu + \frac{\eta_{55}}{c_5^2} \partial_\tau^2 \right) \bar{h}_{\alpha\beta}$$

Solve with principal part Green's function

$$\bar{h}_{\alpha\beta}\left(x, \tau \right) = \frac{4G}{c^4} \int d^3 x' \frac{T_{\alpha\beta}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau \right)}{|\mathbf{x}-\mathbf{x}'|}$$

Post-Newtonian Approximation

Point source $X = (cT(\tau), \mathbf{0})$ in co-moving frame where $\dot{T} = 1 + \alpha(\tau)/2$ and $\alpha^2 \approx 0$

$$T^{00} = mc^2 \dot{T}^2 \delta^3(\mathbf{x}) \rho(t - T(\tau)) \qquad T^{\alpha i} = 0 \qquad T^{55} = \frac{c_5^2}{c^2} T^{00} \approx 0$$

Writing $M(\tau) = m \rho(t - T(\tau))$ a slowly varying density function

$$\bar{h}^{00}(x, \tau) = \frac{4GM}{c^2 R} \dot{T}^2 \qquad \bar{h}^{\alpha i}(x, \tau) = 0 \qquad \bar{h}^{55}(x, \tau) = \frac{c_5^2}{c^2} \bar{h}^{00} \approx 0$$

$$\Gamma^{\mu}_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} \qquad \Gamma^{\mu}_{50} = \frac{1}{2c_5} \eta^{\mu 0} \partial_\tau h_{00}$$

Equations of motion for a test particle in spherical coordinates putting $\theta = \pi/2$

$$\ddot{t} = (\partial_\tau h_{00}) \dot{t} + \dot{\mathbf{x}} \cdot (\nabla h_{00}) \dot{t}^2 \approx \frac{2GM}{c^2 R} \left(1 + \frac{\alpha(\tau)}{2}\right) \dot{\alpha}(\tau) \dot{t}$$

$$\ddot{\mathbf{x}} = \frac{c^2}{2} (\nabla h_{00}) \dot{t}^2 \longrightarrow \begin{cases} 2\dot{R}\dot{\phi} + R\ddot{\phi} = 0 \longrightarrow \dot{\phi} = \dfrac{L}{MR^2} \\[2mm] \ddot{R} - \dfrac{L^2}{M^2 R^3} = -\dfrac{GM}{R^2} \dot{t}^2 \dot{T}^2 \end{cases}$$

Post-Newtonian Evolution

Solution to t equation neglecting $\dot{R}/c \ll 1$ and $\partial_\tau \rho \approx 0$

$$\dot{t} = \exp\left[\frac{2GM}{c^2 R}\left(\alpha + \frac{1}{4}\alpha^2\right)\right] \longrightarrow \dot{t}^2 \dot{T}^2 \simeq 1 + \frac{1}{2}\left(1 + \frac{2GM}{c^2 R}\right)\alpha$$

Taking $\alpha = 0$ recovers $\dot{t} = 1$ for Newtonian case

Using solution to t equation in radial equation

$$\frac{d}{d\tau}\left\{\frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{L^2}{M^2 R^2} - \frac{GM}{R}\left(1 + \frac{1}{2}\alpha\left(\tau\right)\right)\right\} = -\frac{GM}{2R}\frac{d}{d\tau}\alpha\left(\tau\right)$$

LHS is $\dfrac{dK}{d\tau}$ where K is particle Hamiltonian = particle mass

In source rest frame, $\dot{T} \neq 1 \Rightarrow \Delta$energy without Δmomentum $\Rightarrow \Delta$mass

Source transfers mass to perturbed metric field $h_{\alpha\beta}$

Particle mass not generally conserved $\longrightarrow$ particle absorbs mass from $h_{\alpha\beta}$

Taking $\alpha = 0$ recovers nonrelativistic motion with conserved Hamiltonian

3+1 Formalism in General Relativity

Time evolution formalism for Einstein equations

Formulate field equations as initial value problem

Decompose 10 components of spacetime metric into:

6 component space metric

3 component (spatial) shift vector

1 component (time) lapse

Decompose 10 components of Einstein equations into:

6 component partial differential equation (PDE) of second order in ∂_t

4 constraints on space metric, its 1st t-derivative, energy-momentum

Based on mathematics of embedded hypersurfaces

Darmois (1927), Lichnerowicz (1939), Choquet-Bruhat (1952)

Intrinsic structure of hypersurface

Extrinsic structure imposed by embedding hypersurface in larger manifold

Arnowitt, Deser, Misner (1962)

ADM Hamiltonian formulation of GR

Intrinsic and extrinsic structure $\longrightarrow$ canonical conjugate field variables

4+1 formalism generalizes 3+1 framework as presented in:

Gourgoulhon (2007), Bertschinger (2005), ADM (1962), Isham (1992), Blau (2020)

Schematic Outline of 3+1 Formalism

Define a foliation of 4D spacetime

    Choose scalar field $t(x)$ on spacetime

    Hypersurface = simultaneous spacetime points x sharing $t(x) = constant$

    Choose $t(x)$ to identify spacelike hypersurfaces (3D space manifold at time t)

        Each vector V tangent to each hypersurface is spacelike ($V^2 > 0$)

        Vector n normal to hypersurface is timelike ($n^2 = -1$)

Projection operators split tangent and normal components of 4D objects

    Induced space metric γ_{ij} is spacelike projection of spacetime metric $g_{\mu\nu}$

    Spacetime metric $g_{\mu\nu} \longrightarrow \{\gamma_{ij}$, (normal) lapse N, and (tangent) shift $N^i\}$

    Spacelike projection of 4D covariant derivative compatible with γ_{ij}

    Spacelike projection of 4D curvature $\longrightarrow$ usual 3D (intrinsic) curvature

    Timelike projection of 4D curvature $\longrightarrow$ extrinsic curvature of t-evolving space

Einstein equations

    Project Einstein equations onto spacelike hypersurface and timelike normal

    10 components of Einstein equations split into two groups

        6 second order PDEs describing t-evolution of space metric γ_{ij}

        4 non-evolving constraints on initial conditions

4+1 Formalism in SHP General Relativity

Schematic Outline I: The Embedding

5D pseudo-spacetime coordinates $X = (x, c_5\tau) \in \mathcal{M}_5 = \mathcal{M} \times R$

   Manifold $\mathcal{M}_5$ an admixture of 4D spacetime geometry and τ-evolution

   In flat pseudo-spacetime $g_{\alpha\beta} \to \eta_{\alpha\beta} = \text{diag}\left(-1, 1, 1, 1, \sigma\right)$ where $\sigma = \pm 1$

Natural foliation of 5D pseudo-spacetime $\mathcal{M}_5 \longrightarrow$ spacetime hypersurface $\sum_\tau$

   Time function $\tau(X) \longrightarrow \sum_{\tau_0} = \left\{ X \in \mathcal{M}_5 \mid S(X) = \tau(X) - \tau_0 = 0 \right\}$

   Vector n normal to hypersurface $\sum_{\tau_0}$ is τ-like ($n^2 = \sigma = \pm 1$)

Projection operators split tangent and normal components of 5D objects

   Induced spacetime metric $\gamma_{\mu\nu}$ is projection onto $\mathcal{M}$ of 5D metric $g_{\alpha\beta}$

   5D metric $g_{\alpha\beta} \longrightarrow \left\{ \gamma_{\mu\nu}, \text{lapse } N, \text{ and (tangent) shift } N^\mu \right\}$

   Spacetime projection of 5D covariant derivative compatible with $\gamma_{\mu\nu}$

   Spacetime projection of 5D curvature $\longrightarrow$ usual 4D (intrinsic) curvature $R_{\mu\nu\lambda\rho}$

   τ projection of 5D curvature $\longrightarrow$ extrinsic curvature of τ-evolution $K_{\alpha\beta}$

5D Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = 0$$

5 constraints on solution to field equations

Project Einstein equations onto spacetime hypersurface and τ-like normal

Decompose 15 components of field equations into two groups

10 unconstrained second order PDEs

Describe τ-evolution of spacetime metric $\gamma_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$

5 non-evolving constraints on initial conditions: $\left\{ \gamma_{\mu\nu}, \partial_\tau \gamma_{\mu\nu}, T_{\alpha\beta} \right\}$

Foliation of 5D Pseudo-Spacetime $\mathcal{M}_5 = \mathcal{M} \times R$

Embedding

Injective mapping $\Phi : \mathcal{M} \to \mathcal{M}_5$ with $\begin{cases} x^\mu \in \mathcal{M}, \ \mu = 0,1,2,3 \\ X^\alpha = (x, c_5\tau) \in \mathcal{M}_5, \ \alpha = 0,1,2,3,5 \end{cases}$

4D hypersurface — implicitly defined submanifold

$\sum_{\tau_0} = \left\{ X \in \mathcal{M}_5 \ \big| \ S(X) = 0 \right\}$ where $S(X) = \tau(X) - \tau_0 = X^5/c_5 - \tau_0$

Rank 4 Jacobian

$E_\mu^\alpha = \left(\dfrac{\partial X^\alpha}{\partial x^\mu} \right)_{\tau_0} \longrightarrow E_\mu = \partial_\mu = \partial/\partial x^\mu$ as basis for tangent space of $\sum_{\tau_0}$

Unit normal to $\sum_{\tau_0}$

$n_\alpha = \sigma \left| g^{55} \right|^{-1/2} \partial_\alpha S(X) \longrightarrow \begin{cases} n \cdot E_\mu = n_\alpha E_\mu^\alpha = 0 \\ n^2 = g^{\alpha\beta} n_\alpha n_\beta = \sigma \end{cases}$

Induced metric on $\mathcal{M}$

$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta = g_{\alpha\beta} \dfrac{\partial X^\alpha}{\partial x^\mu} \dfrac{\partial X^\beta}{\partial x^\nu} dx^\mu dx^\nu = \gamma_{\mu\nu} dx^\mu dx^\nu \longrightarrow \gamma_{\mu\nu} = g_{\alpha\beta} E_\mu^\alpha E_\nu^\beta$

Projection Operators

Projections

$$A \in \mathcal{T}_x\left(\mathcal{M}_5\right) = \text{tangent space of } \mathcal{M}_5 \longrightarrow \left\{ \begin{array}{l} A_\parallel = \sigma\left(A \cdot n\right) n \\ A_\perp = A - \sigma\left(A \cdot n\right) n \end{array} \right.$$

Normal Projection Operator

$$N_{\alpha\beta} = \sigma n_\alpha n_\beta \qquad N_{\alpha\gamma} N^{\gamma\beta} = \sigma^2 n^2 \, n_\alpha n^\beta = N_\alpha^\beta$$

Tangent Projection Operator

$$P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \qquad P_\beta^\alpha = g^{\alpha\beta} - \sigma n^\alpha n^\beta \qquad P_{\alpha\gamma} P^{\gamma\beta} = P_\alpha^\beta = \delta_\alpha^\beta - \sigma n_\alpha n^\beta$$

Completeness Relation

$$\delta_\beta^\alpha = P_\beta^\alpha + \sigma n^\alpha n_\beta$$

Projector $P_{\mu\nu}$ restricted to $\sum_\tau$ is metric $\gamma_{\mu\nu}$

$$V \in \mathcal{T}_x\left(\mathcal{M}_5\right) \longrightarrow V_\perp^\alpha = P_\beta^\alpha V^\beta \in \mathcal{T}_x\left(\Sigma_\tau\right) \longrightarrow \exists\, v \in \mathcal{T}_x\left(\mathcal{M}\right) \text{ such that } V_\perp^\alpha = v^\mu E_\mu^\alpha$$

$$\gamma_{\mu\nu} = g_{\alpha\beta} E_\mu^\alpha E_\nu^\beta = \left(P_{\alpha\beta} + \sigma n_\alpha n_\beta\right) E_\mu^\alpha E_\nu^\beta = P_{\alpha\beta} E_\mu^\alpha E_\nu^\beta = P_{\mu\nu}$$

Decomposition of the Metric

Time evolution

Embedding $X^\alpha(x,\tau) \longrightarrow$ $\begin{cases} \text{hypersurface} & \sum_{\tau_0} = \{X^\alpha \mid \tau(X) = \tau_0\} \\ \text{trajectory} & X^\alpha(\tau) = \{X^\alpha(x,\tau) \mid x = x_0\} \end{cases}$

General trajectory: normal component $\in \sum_{\tau_0+\delta\tau}$ and tangent component $\in \sum_{\tau_0}$

$$\tau \to \tau + \delta\tau \ \Rightarrow X^\alpha + \left(\frac{\partial X^\alpha}{\partial \tau}\right)_{x_0} \delta\tau = X^\alpha + \left(Nn^\alpha + N^\mu E_\mu^\alpha\right)\delta\tau$$

Lapse function N and Shift vector N^μ

Spacetime shift

$$X^\alpha \to X^\alpha + \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)_{\tau_0} \delta x^\mu = X^\alpha + E_\mu^\alpha \delta x^\mu$$

5D interval

$$ds^2 = g_{\alpha\beta}dX^\alpha dX^\beta = g_{\alpha\beta}\left[Nn^\alpha c_5 d\tau + E_\mu^\alpha\left(N^\mu c_5 d\tau + dx^\mu\right)\right]\left[Nn^\beta c_5 d\tau + E_\nu^\beta\left(N^\nu c_5 d\tau + dx^\nu\right)\right]$$

Decomposition of metric using $\quad n^2 = \sigma \qquad n_\alpha E_\mu^\alpha = 0 \qquad \gamma_{\mu\nu} = g_{\alpha\beta}E_\mu^\alpha E_\nu^\beta$

$$g_{\alpha\beta} = \left[\begin{array}{cc} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu}N^\mu N^\nu \end{array}\right] \qquad g^{\alpha\beta} = \left[\begin{array}{cc} \gamma^{\mu\nu} + \sigma\dfrac{1}{N^2}N^\mu N^\nu & -\sigma\dfrac{1}{N^2}N^\mu \\ -\sigma\dfrac{1}{N^2}N^\mu & \sigma\dfrac{1}{N^2} \end{array}\right]$$

Intrinsic Geometry

Covariant derivatives — compatibility — curvature

On $\mathcal{M}_5$ $\quad g_{\beta\gamma} \longrightarrow \Gamma_{\sigma\beta\gamma} \longrightarrow \nabla_\alpha g_{\beta\gamma} = 0 \longrightarrow \left[\nabla_\beta, \nabla_\alpha\right] X_\delta = X_\gamma R^\gamma_{\delta\alpha\beta}$

On $\mathcal{M}$ $\quad \gamma_{\mu\nu} \longrightarrow \Gamma_{\mu\nu\lambda} \longrightarrow D_\mu \gamma_{\nu\lambda} = 0 \longrightarrow \left[D_\nu, D_\mu\right] X_\rho = X_\lambda R^\lambda_{\rho\mu\nu}$

Projected Covariant Derivative

For $V \in \mathcal{M}_5$ define $\bar{\nabla}_\alpha V^\perp_\beta = \left(P^\gamma_\alpha \nabla_\gamma\right) \left(P^\delta_\beta V_\delta\right) = P^\gamma_\alpha P^\delta_\beta \nabla_\gamma V_\delta$

using $\bar{\nabla}_\alpha P_{\beta\gamma} = -\sigma P^{\alpha'}_\alpha P^{\beta'}_\beta P^{\gamma'}_\gamma \left(\left(\nabla_{\alpha'} n_{\gamma'}\right) n_{\beta'} + n_{\beta'} \nabla_{\alpha'} n_{\gamma'}\right) = 0$

Associate $\quad D_\alpha = \bar{\nabla}_\alpha$ on $\sum_\tau \longrightarrow D_\mu$ on $\mathcal{M}$

through $\quad D_\mu = E^\alpha_\mu D_\alpha = E^\alpha_\mu P^\gamma_\alpha \nabla_\gamma = E^\gamma_\mu \nabla_\gamma$

Projected Curvature $\bar{R}^\gamma_{\delta\alpha\beta}$

On $\sum_\tau \quad \gamma_{\beta\gamma} \longrightarrow \bar{\Gamma}_{\sigma\beta\gamma} \longrightarrow D_\alpha \gamma_{\beta\gamma} = 0 \longrightarrow \left[D_\beta, D_\alpha\right] X_\delta = X_\gamma \bar{R}^\gamma_{\delta\alpha\beta}$

Extrinsic Geometry
Extrinsic Curvature

Curvature of $\mathcal{M}$ as manifold embedded in $\mathcal{M}_5$ as $\sum_\tau$

Gradient $\nabla_\gamma n_\delta$ in $\mathcal{M}_5$

Extrinsic curvature = projection of gradient onto $\sum_\tau \longrightarrow K_{\alpha\beta} = -P_\alpha^\gamma P_\beta^\delta \nabla_\gamma n_\delta$

$\quad$ Using $n^2 = \sigma \longrightarrow \left(\nabla_\alpha n_\beta\right) n^\beta = 0 \longrightarrow \nabla_\alpha n_\beta \in \sum_\tau \longrightarrow P_\beta^\delta \left(\nabla_\gamma n_\delta\right) = \nabla_\gamma n_\beta$

$K_{\alpha\beta} = -P_\alpha^\gamma \nabla_\gamma n_\beta = -\nabla_\alpha n_\beta + \sigma n_\alpha \left(n^\gamma \nabla_\gamma n_\beta\right)$

$\quad$ Using $n_\beta = \sigma \left|g^{55}\right|^{-1/2} \partial_\beta S\left(X\right) = \sigma N \nabla_\beta S\left(X\right)$

$\quad$ Expanding $n^\gamma \nabla_\gamma n_\beta = n^\gamma \left(\nabla_\gamma N\right) \dfrac{n_\beta}{N} + n^\gamma N \nabla_\beta \left(\dfrac{n_\gamma}{N}\right) = -\sigma \dfrac{1}{N} D_\beta N$

$K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \dfrac{1}{N} D_\beta N$

Extrinsic Geometry
Evolution of Hypersurface $\sum_\tau$

Time Evolution of $\sum_\tau$

$$\delta X^\alpha = \left(\frac{\partial X^\alpha}{\partial x^5}\right)_{x_0} \delta x^5 = \left(\frac{\partial X^\alpha}{\partial \tau}\right)_{x_0} \delta\tau \longrightarrow E_5^\alpha = (\partial_5)^\alpha = Nn^\alpha + N^\mu E_\mu^\alpha$$

Define $m^\alpha = Nn^\alpha \;\Rightarrow\; m^2 = \sigma N^2 \longrightarrow \partial_5 = m + \mathbf{N}$

Using $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \dfrac{1}{N} D_\beta N$

$$\nabla_\beta m_\alpha = N\nabla_\beta n_\alpha + n_\alpha \nabla_\beta N = -NK_{\beta\alpha} - n_\beta D_\alpha N + n_\alpha \nabla_\beta N$$

Lie derivative of $\gamma_{\mu\nu}$ along ∂_5

$$\mathcal{L}_5 = \mathcal{L}_m + \mathcal{L}_\mathbf{N}$$

Using $\mathcal{L}_m \gamma_{\alpha\beta} = m^\gamma \nabla_\gamma \gamma_{\alpha\beta} + \gamma_{\gamma\beta} \nabla_\alpha m^\gamma + \gamma_{\alpha\gamma} \nabla_\beta m^\gamma = -2NK_{\alpha\beta}$

Evolution equation for spacetime metric

$$\mathcal{L}_5\, \gamma_{\alpha\beta} - \mathcal{L}_\mathbf{N}\, \gamma_{\alpha\beta} = -2NK_{\alpha\beta} \longrightarrow \mathcal{L}_5\, \gamma_{\mu\nu} - \mathcal{L}_\mathbf{N}\, \gamma_{\mu\nu} = -2NK_{\mu\nu}$$

Decomposition of the Riemann Tensor
Gauss-Codazzi Relations

Using $\qquad \delta_\alpha^{\alpha'} = P_\alpha^{\alpha'} + \sigma n_\alpha n^{\alpha'} \qquad E_\mu^\alpha P_\alpha^{\alpha'} = E_\mu^{\alpha'} \qquad n_\alpha E_\mu^\alpha = 0 \qquad$ to expand

$$R_{\delta\alpha\beta}^\gamma = \delta_\alpha^{\alpha'} \delta_\beta^{\beta'} \delta_{\gamma'}^\gamma \delta_\delta^{\delta'} R_{\delta'\alpha'\beta'}^{\gamma'} \longrightarrow \begin{cases} E_\mu^\alpha E_\nu^\beta E_\gamma^\lambda E_\sigma^\delta P_\alpha^{\alpha'} P_\beta^{\beta'} P_{\gamma'}^\gamma P_\delta^{\delta'} R_{\delta'\alpha'\beta'\gamma'} = R_{\sigma\mu\nu}^\lambda \\[2mm] E_\mu^\alpha E_\nu^\beta E_\gamma^\lambda P_{\gamma'}^\gamma n^\delta P_\alpha^{\alpha'} P_\beta^{\beta'} R_{\delta\alpha'\beta'}^{\gamma'} = \sigma N R_{5\mu\nu}^\lambda \\[2mm] E^{\alpha\mu} E_\nu^\beta P_{\alpha\alpha'} n^\delta P_\beta^{\beta'} n^\gamma R_{\delta\beta'\gamma}^{\alpha'} = N^2 R_{5\nu5}^\mu \end{cases}$$

where $R_{\delta\alpha\beta}^\gamma n^\delta n^\alpha n^\beta = 0$ because of the symmetries of the Riemann tensor

Gauss relation

Decomposing projected Ricci equation for V tangent to $\sum_\tau$

$$\left[D_\beta, D_\alpha \right] V_\delta = V_\gamma \bar{R}_{\delta\alpha\beta}^\gamma \longrightarrow R_{\nu\lambda\rho}^\mu = \bar{R}_{\nu\lambda\rho}^\mu - \sigma \left(K_\lambda^\mu K_{\rho\nu} - K_\rho^\mu K_{\lambda\nu} \right)$$

Codazzi relation

Decomposing Ricci equation for unit normal n

$$\left[\nabla_\beta, \nabla_\alpha \right] n^\gamma = R_{\gamma'\alpha\beta}^\gamma n^{\gamma'} \longrightarrow R_{\mu\nu\lambda}^5 = \sigma \frac{1}{N} \left(D_\lambda K_{\nu\mu} - D_\nu K_{\lambda\mu} \right)$$

Evolution of the Extrinsic Curvature

Decompose Ricci equation for unit normal n

$$\nabla_\beta \nabla_\gamma n^\alpha - \nabla_\gamma \nabla_\beta n^\alpha = R^\alpha_{\delta\beta\gamma} n^\delta$$

Project once onto n and twice onto $\sum_\tau$

$$P_{\alpha\alpha'} n^{\gamma'} P_\beta^{\beta'} \left(\nabla_{\beta'} \nabla_{\gamma'} n^{\alpha'} - \nabla_{\gamma'} \nabla_{\beta'} n^{\alpha'} \right) = P_{\alpha\alpha'} n^{\gamma'} P_\beta^{\beta'} R^{\alpha'}_{\delta\beta'\gamma'} n^\delta$$

Use $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \dfrac{1}{N} D_\beta N$

to obtain $P_{\alpha\alpha'} n^{\gamma'} P_\beta^{\beta'} R^{\alpha'}_{\delta\beta'\gamma'} n^\delta = -K_{\alpha\gamma} K^\gamma_{\ \beta} + \dfrac{1}{N} D_\beta D_\alpha N + P^\gamma_{\ \alpha} P^\delta_{\ \beta} n^\varepsilon \nabla_\varepsilon K_{\gamma\delta}$

Lie derivative of $K_{\alpha\beta}$

$$\left(\mathcal{L}_5 - \mathcal{L}_{\mathbf{N}} \right) K_{\alpha\beta} = \mathcal{L}_m K_{\alpha\beta} = m^\gamma \nabla_\gamma K_{\alpha\beta} + K_{\gamma\beta} \nabla_\alpha m^\gamma + K_{\alpha\gamma} \nabla_\beta m^\gamma$$

Using $\nabla_\beta m^\alpha = -N K^\alpha_\beta - n_\beta D^\alpha N + n^\alpha \nabla_\beta N$

and combining the above leads to

$$P^{\alpha'}_{\ \alpha} P^{\beta'}_{\ \beta} R_{\ \alpha'\beta'} = \sigma \dfrac{1}{N} \mathcal{L}_m K_{\alpha\beta} + \sigma \dfrac{1}{N} D_\alpha D_\beta N + \bar{R}_{\alpha\beta} - \sigma K K_{\alpha\beta} + \sigma 2 K^\delta_\alpha K_{\beta\delta}$$

Decomposition of the Einstein Equations

Evolution Equation for $K_{\alpha\beta}$

Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta} \longrightarrow R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) \qquad T = g^{\alpha\beta}T_{\alpha\beta}$$

Decompose $T_{\alpha\beta}$

$$T_{\alpha\beta} = T_{\alpha'\beta'}\left(P_{\alpha}^{\alpha'} + \sigma n^{\alpha'}n_{\alpha}\right)\left(P_{\beta}^{\beta'} + \sigma n^{\beta'}n_{\beta}\right) = S_{\alpha\beta} + 2\sigma n_{\alpha}p_{\beta} + n_{\alpha}n_{\beta}\kappa$$

where

$$S_{\alpha\beta} = P_{\alpha}^{\alpha'}P_{\beta}^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{\mu\nu} \qquad\qquad p_{\beta} = -n^{\alpha'}P_{\beta}^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{5\mu}$$

$$\kappa = n^{\alpha}n^{\beta}T_{\alpha\beta} \longleftrightarrow T_{55} \qquad\qquad T = S + \sigma\kappa$$

Combining $P_{\alpha}^{\alpha'}P_{\beta}^{\beta'}\left(T_{\alpha'\beta'} - \frac{1}{2}g_{\alpha'\beta'}T\right) = S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\left(S + \sigma\kappa\right)$ with $\mathcal{L}_m K_{\alpha\beta}$ leads to

$$\left(\mathcal{L}_5 - \mathcal{L}_{\mathbf{N}}\right)K_{\mu\nu} = -D_{\mu}D_{\nu}N$$
$$+ N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu}^{\lambda}K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}$$

Decomposition of the Einstein Equations
Constraint Equations

Projecting Einstein equations twice onto normal n

$$\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) n^\alpha n^\beta = \frac{8\pi G}{c^4} T_{\alpha\beta} n^\alpha n^\beta \; \longrightarrow \; R_{\alpha\beta} n^\alpha n^\beta - \frac{1}{2}\sigma R = \frac{8\pi G}{c^4}\kappa$$

Contracting indices in Gauss relation

$$R - 2\sigma R_{\alpha\beta} n^\alpha n^\beta = \bar{R} - \sigma \left(K^2 - K^{\alpha\delta} K_{\alpha\delta} \right)$$

Hamiltonian Constraint

Combining above leads to

$$\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -\sigma \frac{16\pi G}{c^4}\kappa$$

Momentum Constraint

Projecting Einstein equations onto $\sum_\tau$ and normal n

$$n^\alpha P^{\beta'}_{\ \beta} \left(R_{\alpha\beta'} - \frac{1}{2} g_{\alpha\beta'} R \right) = n^\alpha P^{\beta'}_{\ \beta} \frac{8\pi G}{c^4} T_{\alpha\beta'} = -\frac{8\pi G}{c^4} p_\beta$$

Combining with Codazzi relation leads to

$$D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Decomposition of the Einstein Equations
Summary of Differential Equations in 4+1 Formalism

Evolution equation for spacetime metric

$$\frac{1}{c_5} \mathcal{L}_\tau \, \gamma_{\mu\nu} - \mathcal{L}_{\mathbf{N}} \, \gamma_{\mu\nu} = -2NK_{\mu\nu}$$

Evolution equation for extrinsic curvature

$$\left(\frac{1}{c_5} \mathcal{L}_\tau - \mathcal{L}_{\mathbf{N}} \right) K_{\mu\nu} = -D_\mu D_\nu N$$
$$+ N \left\{ -\sigma \bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda} + \sigma \frac{8\pi G}{c^4} \left[S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} \left(S + \sigma \kappa \right) \right] \right\}$$

Hamiltonian Constraint

$$\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -\sigma \frac{16\pi G}{c^4} \kappa$$

Momentum Constraint

$$D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Evolution Versus Constraints

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0$$

In D dimensions symmetric tensor $G^{\alpha\beta}$ has $D(D+1)/2$ components

$G^{\alpha\beta}$ has 15 components on $\mathcal{M}_5$

5 components of Bianchi relation $\longrightarrow$ 10 independent components $+$ 5 constraints

Order of τ derivative

Einstein equations $G_{\alpha\beta} = \dfrac{8\pi G}{c^4} T_{\alpha\beta}$ $\longrightarrow$ 2$^{\text{nd}}$ order in τ derivatives of $g_{\alpha\beta}$

Initial conditions for evolution PDE $= \left\{ g_{\alpha\beta} \, , \partial_\tau g_{\alpha\beta} \, , T_{\alpha\beta} \right\}$

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = 0 \longrightarrow \frac{1}{c_5} \partial_\tau G^{5\beta} = -\nabla_\mu G^{\mu\beta} + \{ \text{Christoffel symbols} \} \times G^{\alpha\beta}$$

RHS at most 2$^{\text{nd}}$ order in ∂_τ $\implies$ $G^{5\beta}$ at most 1$^{\text{st}}$ order in ∂_τ

$G^{5\beta} \longrightarrow$ 5 constraints on initial conditions

Bianchi relation: $G^{\alpha\beta}\Big|_{\tau_0} = 0 \Rightarrow \dfrac{1}{c_5} \partial_\tau \, G^{5\beta}\Big|_{\tau_0} = 0 \longrightarrow$ constraint is conserved

Static Schwarzschild-like Geometry

Metric for $T_{\alpha\beta} = 0 \longrightarrow S_{\mu\nu} = p_\mu = \kappa = 0$

$$ds^2 = -c^2 B dt^2 + A dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \sigma W c_5^2 d\tau^2$$

where

$$B(r) = A^{-1}(r) = 1 - \Phi_0(r) = 1 - \frac{GM}{rc^2} \qquad W = W(x,\tau)$$

$$g_{\alpha\beta} = \left[\begin{array}{cc} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{array} \right] = \left[\begin{array}{cc} \gamma_{\mu\nu}(r) & 0 \\ 0 & \sigma N^2(x,\tau) \end{array} \right]$$

with $\bar{R} = 0$ for Schwarzschild spacetime metric and $N = \sqrt{W}$

Dynamical Equations $\longrightarrow$ 4D wave equation for $\sqrt{W}$

$$(\partial_5 - \mathcal{L}_{\mathbf{N}})\gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow \partial_\tau \gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow K_{\mu\nu} = 0$$

$$\partial_5 K_{\mu\nu} = -D_\mu D_\nu N + N\left(-\sigma \bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda}\right) \longrightarrow \gamma^{\mu\nu} D_\mu D_\nu \sqrt{W(x,\tau)} = 0$$

Constraints trivially satisfied for $\bar{R} = K_{\mu\nu} = p_\nu = \kappa = 0$

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma \frac{16\pi G}{c^4}\kappa \qquad D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

Perturbation Around Schwarzschild Geometry

Admit variable mass

$$M(\tau) = M[1 + \alpha(\tau)] \qquad \alpha^2 \ll 1 \longrightarrow B = A^{-1} = 1 - \Phi_0[1 + \alpha(\tau)]$$

Take $W = 1$

4D connection τ-dependent but retains unperturbed form $\longrightarrow \bar{R} = 0$

Dynamical equations (neglecting terms in α^2 and Φ_0^2)

Metric evolution

$$\partial_5 \gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow K_{\mu\nu} = -\frac{1}{2c_5}\partial_\tau \gamma_{\mu\nu} = -\frac{\Phi_0\dot{\alpha}(\tau)}{2c_5}\text{diag}\left(1, \frac{1}{B^2}, 0, 0\right)$$

$$K_\nu^\mu = \gamma^{\mu\lambda}K_{\lambda\nu} = -\frac{\Phi_0\dot{\alpha}(\tau)}{2c_5}\text{diag}(-1, 1, 0, 0) \longrightarrow K = K_\mu^\mu = 0$$

Curvature evolution, using $\bar{R} = 0$, $N = 1$, $N^\mu = 0$, and $\left(K_{\mu\nu}\right)^2 \propto \alpha^2 \approx 0$

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2c_5^2}\Phi_0\ddot{\alpha}(\tau)\,\text{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}(S + \sigma\kappa)\right]$$

Perturbing Energy Momentum Tensor

Hamiltonian constraint $\longrightarrow$ no perturbing mass density

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa \longrightarrow \kappa \propto \alpha^2\Phi_0^2 \approx 0$$

Momentum constraint $\longrightarrow$ perturbing mass current into r direction

$$p_\nu = D_\mu K_\nu^\mu - D_\nu K = \partial_\mu K_\nu^\mu + K_\nu^\lambda \Gamma_{\lambda\mu}^\mu - K_\lambda^\mu \Gamma_{\nu\mu}^\lambda$$

$$p_0 = \partial_r K_0^1 + K_0^0\Gamma_{0\mu}^\mu - K_\lambda^\mu\Gamma_{0\mu}^\lambda = K_0^0\Gamma_{0\mu}^\mu - K_0^0\Gamma_{00}^0 - K_1^1\Gamma_{01}^1 = 0$$

$$p_1 = \partial_r K_1^1 + K_1^\lambda\Gamma_{\lambda\mu}^\mu - K_\lambda^\mu\Gamma_{1\mu}^\lambda = -\frac{1}{2}\frac{1}{c_5 r}\Phi_0\dot{\alpha}\left(\tau\right)$$

Evolution equation $\longrightarrow$ perturbing energy density and momentum density in r direction

$$\ddot{\alpha}\left(\tau\right) \ \Phi_0 \operatorname{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = -\sigma\frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right]$$

$$S_{00} = S_{11} = \left(-\sigma\frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\right)^{-1}\Phi_0\ddot{\alpha}\left(\tau\right) \qquad S_{22} = S_{33} = S = 0$$

Thank You For Your Patience

Slides and preprints: http://cs.hac.ac.il/staff/martin