Gravitating magnetic monopole via the spontaneous symmetry breaking of pure R^2 gravity

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Pure R^2 gravity

The action of pure R^2 gravity is given by

$$S_0 = \int d^4x \sqrt{-g}\, \alpha\, R^2 \,.$$

Recently discovered that it is

- Invariant under the transformation $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, with $\Box\Omega(x) \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega(x) = 0$. This was dubbed *restricted Weyl invariance*.
- Equivalent to Einstein gravity with non-zero cosmological constant and massless scalar field

Restricted Weyl symmetry

Under a Weyl transformation, $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, we have the following transformations (in four spacetime dimensions)

$$\begin{aligned} \sqrt{|g|} &\to \Omega^4 \sqrt{|g|} \\ R &\to \Omega^{-2} R - 6\,\Omega^{-3} \Box\,\Omega\,. \end{aligned}$$

It follows that under a Weyl transformation we have

$$\sqrt{|g|}\, R^2 \to \sqrt{|g|}\, R^2 - \sqrt{|g|}\, 12\, R\,\Omega^{-1} \Box\Omega + \sqrt{|g|}\, 36\,\Omega^{-2} (\Box\,\Omega)^2\,.$$

Invariant if $\Box\Omega(x) = 0$; this is called restricted Weyl invariance.

Action for the kinetic term of a scalar field is restricted Weyl invariant

We now evaluate how the kinetic term for the scalar field transforms under a Weyl transformation.

$$
\begin{aligned}
&\sqrt{|g|}\, g^{\mu\nu}\, \nabla_\mu \phi\, \nabla_\nu \phi \\
&\to \sqrt{|g|}\, g^{\mu\nu}\, \Omega^2\, \nabla_\mu(\phi\,\Omega^{-1})\, \nabla_\nu(\phi\,\Omega^{-1}) \\
&= \sqrt{|g|}\left(\nabla_\mu \phi\, \nabla^\mu \phi - 2\,\phi\,\Omega^{-1}\, \nabla_\mu \phi\, \nabla^\mu \Omega + \phi^2\,\Omega^{-2}\, \nabla_\mu \Omega\, \nabla^\mu \Omega\right) \\
&= \sqrt{|g|}\Big(\nabla_\mu \phi\, \nabla^\mu \phi - \nabla_\mu(\phi^2 \nabla^\mu(\ln \Omega)) + \phi^2 \nabla_\mu \nabla^\mu (\ln \Omega) + \phi^2 \Omega^{-2} \nabla_\mu \Omega\, \nabla^\mu \Omega\Big) \\
&= \sqrt{|g|}\left(\nabla_\mu \phi\, \nabla^\mu \phi + \phi^2\, \Omega^{-1}\, \Box\, \Omega - \nabla_\mu(\phi^2\, \nabla^\mu(\ln \Omega))\right). \qquad (1)
\end{aligned}
$$

The last term is a total derivative. Therefore, the action for the kinetic term of a scalar field is invariant if $\Box\Omega(x) = 0$ i.e. it is restricted Weyl invariant. In particular, there is no need to add the term $\frac{1}{6}\, R\,\phi^2$ to make it restricted Weyl invariant.

Composition law of restricted Weyl transformations in four dimensions

Consider the two consecutive restricted Weyl transformations in general *d* space-time dimensions

$$\begin{aligned}\tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu}\\ \tilde{\tilde{g}}_{\mu\nu} &= \tilde{\Omega}^2 \tilde{g}_{\mu\nu} = \tilde{\Omega}^2\Omega^2 g_{\mu\nu}\end{aligned} \tag{2}$$

where $\Box\Omega = 0$ and $\tilde{\Box}\tilde{\Omega} = 0$.

In d=4 dimensions (and only d=4) one can show that $\Box(\tilde{\Omega}\Omega) = 0$ so that consecutive restricted Weyl transformations obey a composition law.

Restricted Weyl invariant R^2 action with Higgs field and its equivalent Einstein action

We begin with an action that includes pure R^2 gravity, a non-minimally coupled massless triplet Higgs field Φ and $SU(2)$ non-abelian gauge fields A^i_μ:

$$S_a = \int d^4x\sqrt{-g}\left(\alpha R^2 - \xi R|\vec{\Phi}|^2 - D_\mu\vec{\Phi}D^\mu\vec{\Phi} - \lambda|\vec{\Phi}|^4 + \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\right)$$

where α, ξ and λ are free dimensionless parameters and D_μ is the usual covariant derivative with respect to the non-abelian gauge symmetry. The above action is restricted Weyl invariant i.e. it is invariant under the transformation

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \quad , \quad \vec{\Phi} \to \vec{\Phi}/\Omega \quad , \quad A^i_\mu \to A^i_\mu \;\text{ with } \Box\Omega = 0$$

Equivalent form of the action

Introducing the auxiliary field φ, we can rewrite the above action into the equivalent form

$$S_b = \int d^4x\sqrt{-g}\Big[-\alpha(c_1\varphi + R + \frac{c_2}{\alpha}|\vec{\Phi}|^2)^2 + \alpha R^2 - \xi R|\vec{\Phi}|^2 - D_\mu\vec{\Phi}D^\mu\vec{\Phi} - \lambda|\vec{\Phi}|^4 + \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\Big]$$

where c_1 and c_2 are arbitrary constants.

Expanding the above action we obtain

$$S_c = \int d^4x\sqrt{-g}\Big(- c_1^2\,\alpha\,\varphi^2 - 2\alpha c_1\varphi R - (\xi + 2c_2)R|\vec{\Phi}|^2$$
$$- D_\mu\vec{\Phi}D^\mu\vec{\Phi} - 2c_1c_2\varphi|\vec{\Phi}|^2 - (\alpha^{-1}c_2^2 + \lambda)|\vec{\Phi}|^4 + \frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\Big)\ .$$

The above action is equivalent to the original action and is restricted Weyl invariant as long as φ transforms accordingly; it is invariant under the transformations $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\varphi \to \varphi/\Omega^2$, $\vec{\Phi} \to \vec{\Phi}/\Omega$, $A^i_\mu \to A^i_\mu$ with $\Box\Omega = 0$.

Conformal transformation and Einstein action

After performing the conformal (Weyl) transformation

$$g_{\mu\nu} \to \varphi^{-1} g_{\mu\nu} \quad , \quad \vec{\Phi} \to \varphi^{1/2} \vec{\Phi} \quad , \quad A^i_\mu \to A^i_\mu$$

the above action reduces to an Einstein-Hilbert action with a massive Higgs term plus other terms

$$S_d = \int d^4x \sqrt{-g} \Big(-\alpha c_1^2 - 2\,\alpha c_1 R - D_\mu \vec{\Phi} D^\mu \vec{\Phi} - 2\, c_1 c_2 |\vec{\Phi}|^2 - (\alpha^{-1} c_2^2 + \lambda) |\vec{\Phi}|^4$$
$$- (\xi + 2c_2) R |\vec{\Phi}|^2 + 3\,\alpha c_1 \frac{1}{\varphi^2} \partial_\mu \varphi\, \partial^\mu \varphi +$$
$$(6(\xi + 2c_2) - 1) \varphi^{1/2} \Box \varphi^{-1/2} |\vec{\Phi}|^2 + \frac{1}{4} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \Big).$$

By defining $\psi = \sqrt{-3\,\alpha c_1} \ln \varphi$ the kinetic term for φ can be expressed in the canonical form $-\partial_\mu \psi\, \partial^\mu \psi$.

Spontaneous symmetry breaking

- The conformal (Weyl) transformation is not valid for $\varphi = 0$.
- The vacuum has $\varphi \neq 0$. This vacuum is not invariant under $\varphi \to \varphi/\Omega^2 \Rightarrow$ restricted Weyl symmetry is spontaneously broken. This is evident from the fact that the final action has a massive Higgs and Einstein-Hilbert term.
- The massless scalar ψ is identified as the Nambu-Goldstone boson associated with the broken symmetry.
- It is well known that in spontaneously broken theories the original symmetry is still realized as a shift symmetry of the Goldstone bosons. This is the case here. The final action is invariant under $\varphi \to \varphi/\Omega^2$, $g_{\mu\nu} \to g_{\mu\nu}$, $\vec{\Phi} \to \vec{\Phi}$ with condition $\Box\Omega - \partial_\mu(\ln\varphi)\partial^\mu\Omega = 0$. The Goldstone boson ψ therefore undergoes the shift symmetry $\psi \to \psi - 2\sqrt{-3\,\alpha c_1}\,\ln\Omega$.

Vacuum solutions: Higgs VEV and corresponding Ricci scalar

Unbroken gauge sector

The $\vec{\Phi} = 0$ vacuum with no spontaneous breaking of gauge symmetry yields a cosmological constant of $\Lambda = -c_1/4$ in the final action. The Ricci scalar is then given by

$$R = 4\,\Lambda = -c_1\,.$$

The possible background spacetimes are then de Sitter space if $c_1 < 0$ and anti-de Sitter space for $c_1 > 0$. The constant c_1 cannot be identically zero and Minkowski space is not a valid background here. This supports black holes in either dS or AdS spacetime only and no magnetic monopoles.

Broken gauge sector

- The case with spontaneous breaking of gauge symmetry yields the vacuum solution

$$|\vec{\Phi}|^2 = \frac{-\alpha\, c_1\, \xi}{\xi\, c_2 - 2\alpha\lambda} \quad ; \quad R = \frac{2\alpha\, c_1\, \lambda}{\xi\, c_2 - 2\alpha\lambda}\,.$$

- The VEV of $\vec{\Phi}$ breaks the SU(2) gauge symmetry down to $U(1)$, and this breaking pattern leads to the existence of the monopole.
- A crucial point is that when $\lambda = 0$, we obtain an $R = 0$ vacuum corresponding to a Minkowski background with $|\vec{\Phi}|^2 = \frac{-\alpha\, c_1}{c_2}$ where positivity implies $c_2 > 0$. In contrast to the Minkowski background solution that exists in the original action with $\vec{\Phi} = 0$, the $\vec{\Phi} \neq 0$ Minkowski solution is a perfectly viable background and we know linearizations lead to gravitational waves since it is nothing other than the Minkowski space of Einstein gravity.

Equations of motion for static spherical symmetry: magnetic monopole solutions

We now look for static spherically symmetric solutions. We set $\varphi = 1$. The ansatz for the metric is

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \ .$$

Let us make the ansatz for the gauge field

$$A^{ia} = q(r)\epsilon^{aik}x^k$$

and the Higgs field

$$\vec{\Phi} = f(r)\frac{\vec{x}}{r} \ .$$

Defining

$$1 + r^2 q(r) = a(r)$$

we obtain the following Einstein-Yang-Mills-Higgs (EYMH) action

EYMH action with non-minimal coupling

$$S = \int d^4x\sqrt{-g}(\tilde{\Lambda} + \tilde{\alpha}R - \tilde{\xi}R\vec{\Phi}^2 - D_\mu\vec{\Phi}D^\mu\vec{\Phi} - \tilde{m}^2\vec{\Phi}^2 - \tilde{\lambda}(\vec{\Phi}^2)^2 - \frac{1}{4g^2}F^2_{\mu\nu}$$

$$= 4\pi\int drdt\sqrt{B/A}\,r^2\Big[\tilde{\Lambda} - \tilde{m}^2 f^2 - A(f')^2 - \frac{2a^2f^2}{r^2} - \tilde{\lambda}f^4 - \frac{(a^2-1)^2}{2g^2r^4} - \frac{A(a')^2}{g^2r^2} + (\tilde{\alpha} - \tilde{\xi}f^2)\Big(\frac{2}{r^2} - \frac{2A}{r^2} - \frac{2A'}{r} - \frac{2AB'}{rB} - \frac{A'B'}{2B} + \frac{A(B')^2}{2B^2} - \frac{AB''}{B}\Big)\Big]$$

where for convenience we introduced the new parameters

$$\tilde{\Lambda} = -\alpha c_1^2 \quad ; \quad \tilde{\alpha} = -2\alpha c_1 \quad ; \quad \tilde{\xi} = \xi + 2c_2 \quad ; \quad \tilde{m}^2 = 2c_1c_2\,;\,\tilde{\lambda} = \lambda + c_2^2/\alpha\,.$$

EYMH magnetic monopoles: numerical solutions

We seek gravitating magnetic monopole numerical solutions to the equations of motion in flat, dS and AdS backgrounds with non-minimal coupling constant $\tilde{\xi} = 1/6$. By definition, magnetic monopoles are non-singular at the origin $r = 0$ and have a field configuration of a point-like magnetic charge at large distances. This requires in total five boundary conditions at $r = 0$ and "infinity":

$$a(0) = 1 \quad ; \quad A(0) = 1 \quad ; \quad f(0) = 0 \quad ; \quad a(\infty) = 0 \quad ; \quad f(\infty) = 1 \,.$$

Figure: Monopole in flat background with non-minimal coupling. This case corresponds to $k = 0$ (where $k = -\Lambda/3$).

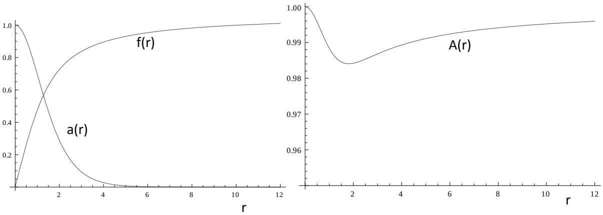


Figure: Monopole in AdS background with non-minimal coupling. The cosmological constant is chosen to be $\Lambda = -1$ (hence $k = 1/3$).

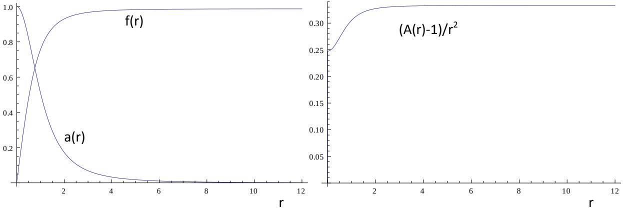
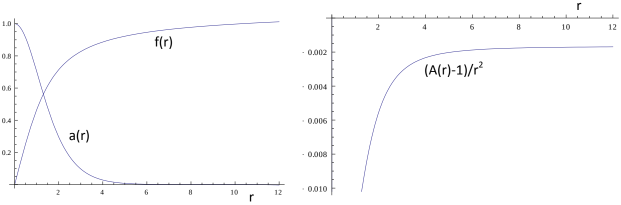


Figure: Monopole in dS background with non-minimal coupling. The cosmological constant is $\Lambda = +0.005$ (hence $k = -0.001667$).



Thank You